Extremal Bipartite Matrices

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ABSTRACT

Let G be an undirected graph on vertices $\{1,\dots,n\}$. Let $M(G)$ be the convex cone of all positive semidefinite hermitian matrices A satisfying $a_{ij}=0$ if (i,j) is not an edge of G. In the case that G is a complete bipartite graph, we characterize all extreme rays of $M(G)$. In doing so, we shall also find the maximum rank that an extreme matrix in $M(G)$ can have.

I. INTRODUCTION

Let G be an undirected graph on vertices $\{1,\dots,n\}$. We let $E(G)$ be the set of edges of G. Let H_n be the set of all complex positive semidefinite hermitian matrices, and let $M(G)$ be the set of all A in H_n satisfying $a_{ij}=0$ if (i,j) is not in $E(G)$. No requirements are placed on the main diagonal entries. Clearly $M(G)$ is a convex cone. The purpose of this paper is to characterize the extreme rays of $M(G)$ in the case that G is a complete bipartite graph. We shall also investigate the ranks of the matrices belonging to these extreme rays.

The reader will note that there are really two convex cones in H_n that we may associate to G, namely $M(G)$ and $M(G)\cap\mathbf{R}$, the real symmetric matrices A satisfying $a_{ij}=0$ if (i,j) is not in $E(G)$. Obviously the extreme

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rays may differ in the two cases, and hence, while we shall refer generically to $M(G)$, we shall distinguish the real and complex cases when appropriate.

Some specific examples of $M(G)$ may be of interest. If $G = K_n$, the complete graph on $\{1,\dots,n\}$, then $M(G)$ is all of H_n. If G is the path from 1 to n, then $M(G)$ is all tridiagonal matrices in H_n. If G is a simple circuit, then $M(G)$ is all tridiagonal matrices, except that the $(1,n)$ and $(n,1)$ entries do not have to be zero. If G is a complete bipartite graph with $E(G) = \{(i,j)|1 \leqslant i \leqslant m,\ m+1 \leqslant j \leqslant n\}$, then $M(G)$ consists of all matrices in H_n of the form

$$\begin{bmatrix} D_1 & B \\ B^* & D_2 \end{bmatrix}$$

where D_1 and D_2 are diagonal matrices of sizes m, $n-m$ respectively.

Part of the motivation for study of $M(G)$ is the so-called matrix completion problem [4, 5]. Here, we have a graph G and a matrix A in which the diagonal entries are specified, $a_{ij} = \bar{a}_{ji}$ is specified if $(i,j) \in E(G)$, and other spaces are left blank. One is concerned with the inertia possibilities when A is completed to a hermitian matrix by filling in the blanks. The connection between the matrix completion problem and the classification of extreme rays in $M(G)$ is not well understood. It seems likely that there is some relation because of the appearance, for example, of chordal graphs (no simple k-circuits for $k > 3$) in both [5] and [2]. The reader is referred to these two papers for a further discussion.

One should note that if it is possible to complete A to a member of H_n, then the set of all possible completions of A in H_n is a convex subset of H_n. Convex sets in H_n arise in many circumstances. We mention one other. If $A \in H_n$ is given, let $S(A)$ be the set of all matrices B in H_n satisfying $A - B \in H_n$. Then $S(A)$ is convex, and A is an extreme point of $S(A)$. The extreme points of $S(A)$ are discussed in [1].

Research on extreme matrices of $M(G)$ first appears in [2]. The authors showed, for example, that G is chordal, i.e., has no simple circuits of length > 3, if and only if every extreme matrix in $M(G)$ has rank 1. This is valid in both the real and the complex case. In a followup paper [3], it was verified that the largest possible rank of an extreme matrix in $M(G)$ is $n-2$, and this is achievable only if G is a simple circuit. There was also a partial solution of the bipartite graph problem, which we will finish in this manuscript.

The following result was proved in [2]. It has been the primary tool used in both [2] and [3]. Although we do not use it here (we resort to more conventional methods), its significance is such that it is worth repeating. It is a test for a matrix in $M(G)$ to be extreme.

THEOREM 1.1. *Let $A \in M(G)$ have rank k. Factor $A = BB^*$ (BB^T in the real case) with $B\, n \times k$. Let $w_1, \dots, w_n$ be the columns of B^*. Clearly $\langle w_i, w_j \rangle = 0$ if (i, j) is not in $E(G)$. For each pair (i, j) not in $E(G)$, let*

$$H_{ij} = w_i w_j^* \qquad \textit{in the complex case},$$

$$S_{ij} = w_i w_j^T + w_j w_i T \qquad \textit{in the real case}.$$

Note that trace $H_{ij} =$ trace $S_{ij} = 0$. *Also observe that we include both H_{ij} and H_{ji} in the complex case. Then A is extreme in $M(G)$ if and only if*

(i) Span$\{H_{ij}\}$ = *all complex matrices of trace* 0 *in the complex case*;
(ii) Span$\{S_{ij}\}$ = *all real symmetric matrices of trace* 0 *in the real case.*

Theorem 1.1 produces an ad hoc method of constructing (in some cases) extreme matrices in $M(G)$ of large rank. We are still frustrated by the lack of an easily implemented algorithm for determining the ranks of extreme matrices in $M(G)$. Finding such an algorithm is not a numerical problem; instead, we probably need to understand more about the ranks of extreme matrices in $M(H)$ where H is a subgraph of G. Until such an algorithm appears, it will be difficult to test related ideas.

II. STATEMENT OF RESULTS

Let $K_{m,n}$ be the complete bipartite graph on vertices $\{1, \dots, m+n\}$, i.e., $E(K_{m,n}) = \{(i, j) | 1 \leqslant i \leqslant m < j \leqslant n\}$. The main purpose of this paper is to determine the maximum rank an extreme matrix in $M(K_{m,n})$ can have. We shall do this in both the real and complex cases. In doing so, we shall characterize all extreme matrices in $M(K_{m,n})$. (By extreme matrix, we mean any matrix in an extreme ray.) We always assume that $m \leqslant n$.

THEOREM 2.1. *In the real case, the maximum rank of any extreme matrix in $M(K_{m,n})$ is*

(i) n *if* $n \leqslant (m^2 - m)/2 + 1$,
(ii) $(m^2 - m)/2 + 1$ *if* $(m^2 - m)/2 + 1 \leqslant n$.

THEOREM 2.2. *In the complex case, the maximum rank of any extreme matrix in $M(K_{m,n})$ is*

(i) n *if* $n \leqslant m^2 - m + 1$,
(ii) $m^2 - m + 1$ *if* $m^2 - m + 1 \leqslant n$.

The following result (in the real or the complex case) is a consequence of the proofs of Theorems 2.1 and 2.2.

THEOREM 2.3. *Let M be the maximum rank an extreme matrix in $M(K_{m,n})$ can have, as given in Theorems 2.1 and 2.2. Then for every integer q between 1 and M, there are extreme matrices of rank q.*

Theorem 2.3 contrasts with the following result in [2]. If G is a simple circuit on n vertices, then in the real case an extreme matrix in $M(G)$ must have rank $n-2$ or 1.

III. SOME REDUCTIONS

To simplify notation, we let $M_{m,n} = M(K_{m,n})$. Let $C_{m,n}$ be the convex compact subset of $M_{m,n}$ consisting of all matrices in $M_{m,n}$ with main diagonal entries all 1. If $A \in C_{m,n}$ and A is extreme in $M_{m,n}$, then A is also extreme in $C_{m,n}$, but the converse is, of course, not necessarily true. The following is an example of the failure of the converse:

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

shows that the matrix on the left is not extreme in $M_{1,2}$ while it is in $C_{1,2}$.

The extreme matrices in $C_{m,n}$ are easy to classify.

LEMMA 3.1. *Let $m \leqslant n$. Let*

$$A = \begin{bmatrix} I_m & B \\ B^* & I_n \end{bmatrix} \tag{3.1}$$

be in $C_{m,n}$. Then A is extreme in $C_{m,n}$ if and only if $BB^* = I_m$, i.e., if and only if the singular values of B are all 1.

Proof. Let U and V be unitary matrices of sizes m and n respectively. Then congruence with $U \oplus V$ maps $C_{m,n}$ onto itself and preserves extreme points. Thus we assume that B has the form

$$\begin{bmatrix} s_1 & 0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 \\ 0 & s_2 & 0 & \cdot & \cdot & \cdot & \cdot & \cdot & 0 \\ \vdots & & & & & & & & \vdots \\ 0 & \cdot & \cdot & \cdot & 0 & s_m & 0 & \cdots & 0 \end{bmatrix}$$

where the s_i are nonnegative. If any $s_i > 1$, then A is not positive semidefinite. If, say, $s_1 < 1$, Let A_ε be the matrix obtained from A by replacing s_1 with $s_1 + \varepsilon$, and let $A_{-\varepsilon}$ be obtained from A by replacing s_1 with $s_1 - \varepsilon$. If $\varepsilon > 0$ is small enough, then A_ε and $A_{-\varepsilon}$ are in $C_{m,n}$ and $2A = A_\varepsilon + A_{-\varepsilon}$, so A is not extreme. Sufficiency is clear, because $C_{m,n}$ must have extreme points and thus the extreme points must occur exactly in the case $s_1 = \cdots = s_m = 1$. This completes the proof. ■

Lemma 3.1 can be used relative to the matrix completion problem. Let A_m and A_n be two given positive definite matrices, and consider all matrix completions for

$$\begin{bmatrix} A_m & ? \\ ? & A_n \end{bmatrix}.$$

Obviously completions exist. Let $L_{m,n}$ be the (convex) set of all such completions. Clearly, $L_{m,n}$ is congruent to $C_{m,n}$, and since congruence preserves extreme points, Lemma 3.1 yields the extreme points of $L_{m,n}$.

We now return to $M_{m,n}$ let

$$A = \begin{bmatrix} D_1 & B \\ B^* & D_2 \end{bmatrix}$$

be in $M_{m,n}$. Thus D_1 and D_2 are real diagonal matrices. If D_1 has r nonzero entries and D_2 has t nonzero entries, we say that A is of *type* (r,t). If A is of type (r,t) and we write $A = A_1 + A_2$ with A_1 and A_2 in $M_{m,n}$, then if row and column i of A are zero, the same must be true of row and column i of A_1 and A_2. Thus A_1 and A_2 must be of type (r_1, t_1) and (r_2, t_2) with $(r_i, t_i) \leqslant (r,t)$.

If A is of type (r,t) let $\tilde{A}$ be the $(r+t)\times(r+t)$ submatrix of A obtained by deleting all zero rows and columns of A. Then $\tilde{A} \in M_{r,t}$, and it is clear that A is extreme in $M_{m,n}$ if and only if $\tilde{A}$ is extreme in $M_{r,t}$. Thus it suffices to determine which matrices of type (m,n) are extreme in $M_{m,n}$. Finally, we observe that $M_{m,n}$ is invariant under nonsingular diagonal congruence, and thus if D is $(m+n)\times(m+n)$ nonsingular diagonal, A is extreme in $M_{m,n}$ if and only if D^*AD is extreme in $M_{m,n}$. Any member of $M_{m,n}$ of type (m,n) is congruent to a member of $C_{m,n}$. Therefore, the proofs of Theorems 2.1 and 2.2 are reduced to finding which matrices in $C_{m,n}$ are extreme in $M_{m,n}$.

IV. PRELIMINARY PROOFS

Let A be a matrix in $C_{m,n}$ of the form (3.1). We need conditions on B in order that A be extreme in $M_{m,n}$. Since $m\leqslant n$, we use Lemma 3.1 to conclude that it is necessary that $BB^*=I_m$. This already shows that any extreme matrix of the form (3.1) must have rank n, and also shows that a extreme matrix in $M_{m,n}$ of rank t must be of type (r,t) for some $(r,t)\leqslant(m,n)$.

LEMMA 4.1. *Suppose A has the form* (3.1) *with $BB^*=I_m$. Then A is extreme in $M_{m,n}$ if and only if BDB^* is never diagonal for any nonscalar real diagonal matrix D.*

Proof. If A is not extreme in $M_{m,n}$, there must be a hermitian matrix of the form

$$H=\begin{bmatrix} D_m & E \\ E^* & D_n \end{bmatrix}$$

with D_m and D_n real diagonal such that for some $\varepsilon>0$, $A\pm\varepsilon H$ is positive semidefinite and H is not a multiple of A. For this to occur, we need the null space of H to contain the null space of A. Now the null space of A consists of all column vectors of the form

$$v=\begin{bmatrix} x \\ -B^*x \end{bmatrix},$$

where x is arbitrary in $\mathbf{C}^m$ ($\mathbf{R}^m$ in the real case). For v to be in the null space of H for all x, we require that $D_m=EB^*$ and $E^*=D_nB^*$. It follows that $D_m=BE^*=BD_nB^*$. If D_n were scalar, say $D_n=\alpha I_n$, then $BB^*=I_m$ implies that $D_m = \alpha I_n$. Now α must be real, and hence $E^* = \alpha B^*$ and $H = \alpha A$. Thus $H \neq \alpha A$ is equivalent to D_n not being scalar. Therefore, if A is not extreme in $M_{m,n}$, the condition of Lemma 4.1 cannot hold.

Conversely, suppose the condition of Lemma 4.1 does not hold. Let D_m and D_n be real nonscalar diagonal matrices such that $BD_nB^* = D_m$. Put

$$H = \begin{bmatrix} BD_nB^* & BD_n \\ D_nB^* & D_n \end{bmatrix}.$$

Since D_n is not scalar, $H \neq \alpha A$. It is also clear that the null space of H contains the null space of A. Thus for $\varepsilon > 0$ small enough, $A \pm \varepsilon H$ is a member of $M_{m,n}$, and Lemma 4.1 is proved. ∎

REMARK 4.2. Lemma 4.1 allows us to provide a striking example of a convex cone with a nonextreme matrix which is the limit of a sequence of extreme matrices. Let $m = n = 2$. For $p = 3,4,\ldots$ let

$$B_p = \begin{bmatrix} \cos(2\pi/p) & \sin(2\pi/p) \\ -\sin(2\pi/p) & \cos(2\pi/p) \end{bmatrix}.$$

Let

$$A_p = \begin{bmatrix} I_2 & B_p \\ B_p^T & I_2 \end{bmatrix}.$$

The reader may easily use Lemma 4.1 to verify for $p = 3,4,\ldots$ that A_p is extreme in $M_{2,2}$ but the limit matrix

$$\begin{bmatrix} I_2 & I_2 \\ I_2 & I_2 \end{bmatrix}$$

is not extreme.

Remark 4.2 indicates a relatively simple characterization of extreme matrices of type (n,n) in $M_{n,n}$.

LEMMA 4.3. *Let*

$$A = \begin{bmatrix} I_n & B \\ B^* & I_n \end{bmatrix}$$

be a member of $C_{n,n}$. Assume that B is unitary, so that A is extreme in $C_{n,n}$ by Lemma 3.1, and hence is a candidate to be an extreme matrix in $M_{n,n}$. Then A is extreme in $M_{n,n}$ if and only if B is irreducible in the following sense: there do not exist permutation matrices P and Q such that PBQ is a nontrivial direct sum.

Proof. Suppose P and Q are permutation matrices such that $PBQ^T = U \oplus V$. Set $H = (P \oplus Q)A(P \oplus Q)^T$. Then A is extreme in $M_{n,n}$ if and only if H is extreme in $M_{n,n}$. However,

$$H = \begin{bmatrix} I_n & U \oplus V \\ (U \oplus V)^* & I_n \end{bmatrix},$$

and hence H decomposes as

$$\begin{bmatrix} I & 0 & U & 0 \\ 0 & 0 & 0 & 0 \\ U^* & 0 & I & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & I & 0 & V \\ 0 & 0 & 0 & 0 \\ 0 & V^* & 0 & I \end{bmatrix},$$

and thus H is not extreme.

Conversely, if B is an irreducible unitary matrix, and D_n is a nonscalar real diagonal matrix, then if BD_nB^* is a diagonal matrix, say E_n, we have $E_n = P^T D_n P$ for some permutation matrix P. It follows that PB commutes with D_n. Since D_n is not scalar, PB (and hence B) is reducible. ∎

REMARK 4.4. The assumption that $m = n$ in Lemma 4.3 is essential, because BD_nB^* must be a (unitary) similarity.

COROLLARY 4.5. *There are extreme matrices of type (n, n) in $M_{n,n}$.*

Proof. Let $B = I_n - (2/n)J_n$, where all entries in J_n are 1. Then B is unitary with no zero entries. Hence

$$A = \begin{bmatrix} I_n & B \\ B^* & I_n \end{bmatrix}$$

is extreme. ∎

V. THE REAL CASE

To this point, the results verified in Sections III and IV have been valid in both the real and the complex case. In this section, all matrices will be real. We now verify Theorem 2.1 (the real case) in a series of Lemmas.

LEMMA 5.1. *If $n > m(m-1)/2+1$, then $M_{m,n}$ has no extreme matrices of type (m,n).*

Proof. Let A be a real matrix of the form (3.1) with $BB^T = I_m$, so that A is a candidate for an extreme point. Let $D_n = \operatorname{diag}(x_1,\ldots,x_n)$ be a diagonal matrix with the x_i real variables. For a given $m \times n$ real matrix B, the requirement that BD_nB^T be diagonal consists of $m(m-1)/2$ homogeneous linear equations in n unknowns. Obviously, $D_n = I_n$ is one solution. Thus we are guaranteed a nonscalar solution for D_n, since $n > m(m-1)/2+1$. The result now follows from Lemma 3.1. ∎

LEMMA 5.2. *Suppose $M_{m,n}$ has an extreme matrix of type (m,n) (i.e., of rank n) and that $m < n$. Then $M_{m+1,n}$ has an extreme matrix of type $(m+1,n)$.*

Proof. Let B be an $m \times n$ real matrix satisfying $BB^T = I_m$ and such that BD_nB^T is never diagonal for any nonscalar diagonal matrix D_n. The rows of B are orthonormal, and since $m < n$, we let B_1 be the $(m+1)\times n$ matrix obtained from B by placing a unit row vector v orthogonal to the rows of B above B. Thus B_1 has the form

$$\begin{bmatrix} v \rightarrow \\ B \end{bmatrix},$$

and clearly $B_1B_1^T = I_{m+1}$. Suppose $D_n = \operatorname{diag}(x_1,\ldots,x_n)$. Then BD_nB^T is a principal submatrix of $B_1 D_n B_1^T$. Thus, if D_n is not scalar, $B_1 D_n B_1^T$ cannot be diagonal. ∎

Lemma 5.2 implies that we need only investigate the existence of extreme points in $M_{m,n}$ of type (m,n) when $n = m(m-1)/2$. The next lemma will therefore complete the proof of Theorem 2.1.

LEMMA 5.3. *Suppose that $n = m(m-1)/2$. Then there is a real $m \times n$ matrix B satisfying $BB^T = mI_m$ and such that if D_n is any nonscalar real diagonal matrix, then BD_nB^T is not diagonal.*

Proof. We construct B explicitly. Lexicographically order the $m(m-1)/2$ pairs of integers chosen from $\{1,\dots,m\}$. If (p,q), $p<q$, is the jth pair, then column j of B contains a 1 in row p, a -1 in row q, and zeros elsewhere. This produces an $m \times (n-1)$ matrix. Complete this to an $m \times n$ matrix B by letting the last column of B consist entirely of 1's.

The following example when $m=3$ and $n=4$ will illustrate. The pairs which concern us are (1,2), (1,3), and (2.3). Thus

$$B = \begin{bmatrix} 1 & 1 & 0 & 1 \\ -1 & 0 & 1 & 1 \\ 0 & -1 & -1 & 1 \end{bmatrix}$$

and $BB^T = 3I_3$.

Consider row k of B. This row contains $k-1$ (-1)'s and $m-k+1$ 1's. Thus every row of B has length $\sqrt{m}$. We next assert that the rows of B are orthogonal. Suppose $p<q$. The last column of B contains all ones. Only one other column of B has nonzero entries in both rows p and q. These entries must be 1 in row p and -1 in row q. Thus rows p and q are orthogonal, and hence $BB^T = mI_m$.

Now Let $D_n = \operatorname{diag}(x_1,\dots,x_n)$ with the x_i real variables, and consider BD_nB^T. We assert that BD_nB^T is not diagonal unless all x_i are the same. The (p,q) entry of BD_nB^T is

$$\sum_{k=1}^{n} b_{pk} x_k b_{qk}.$$

Corresponding to $k=n$, we obtain the term x_k. For k between 1 and $n-1$, there is only one column l such that b_{pl} and b_{ql} are both nonzero. This is the column corresponding to the pair (p,q), and hence we obtain the expression $x_n - x_l$. Obviously, for $s = 1,\dots,n-1$, every term of the form $x_n - x_s$

appears as an off diagonal entry in BD_nB^T, and this implies the conclusion of Lemma 5.2. ∎

Obviously Lemma 5.3 implies Theorem 2.1. In fact, if $(r,t) \leqslant (m,n)$, then $M_{m,n}$ has an extreme matrix of rank t if and only if $t \leqslant m(m-1)/2$. Moreover, any such extreme point must be of type (r,t).

VI. THE COMPLEX CASE

The proof of Theorem 2.2 is similar to that of Theorem 2.1. We are looking for extreme matrices in $M_{m,n}$ of type (m,n). Let $D_n = \operatorname{diag}(x_1,\dots,x_n)$, where the x_i are still real variables. For a given $m \times n$ complex matrix B, the requirement that BD_nB^* be diagonal consists of m^2-m real homogeneous linear equations in n variables. If $n \geqslant m^2-m+2$, there must be a nonscalar solution to this system. Thus, in the complex case, for $n > m^2-m+1$, $M_{m,n}$ has no extreme matrices of type (m,n).

The analogue of Lemma 5.2 is the same in the complex case, so we will move directly to the complex version of Lemma 5.3.

LEMMA 6.1. *Let $n = m^2-m+1$. There is an $m \times n$ complex matrix B satisfying $BB^* = (2m-1)I_m$ and such that if D_n is any nonscalar real diagonal matrix, then BD_nB^* is not diagonal.*

Proof. Fix an angle θ such that $\sin 2\theta \neq 0$. As in Lemma 5.3, lexicographically order the pairs (p,q), $1 \leqslant p < q \leqslant m$. If (p,q) is the kth pair, define columns $2k-1$ and $2k$ of B as follows:

$$b_{p,2k-1} = b_{p,2k} = 1, \qquad b_{q,2k-1} = -e^{i\theta}, \qquad b_{q,2k} = -e^{-i\theta}.$$

Finally, let every entry in column n of B be $\sqrt{2\cos\theta}$. It is easy to check as in Lemma 5.3 that $BB^* = (2m-1)I_m$. We also compute that for $p<q$, the (p,q) entry of BD_nB^* is

$$2\cos\theta\, x_n - e^{i\theta}x_p - e^{-i\theta}x_q,$$

and this value must be zero if BD_nB^* is diagonal. This leads to two real linear equations:

$$2\cos\theta\, x_n - \cos\theta\, x_p - \cos\theta\, x_q = 0,$$

$$-\sin\theta\, x_p + \sin\theta\, x_q = 0.$$

Since $\sin\theta\cos\theta \neq 0$, this implies $x_p = x_q = x_n$. Thus D_n is scalar, and we have proved Lemma 6.1 and Theorem 2.2. ∎

REFERENCES

1 W. Anderson and G. Trapp, The extreme points of a set of positive semidefinite operators, *Linear Algebra Appl* 106:209–217 (1988).

2 J. Agler, J. W. Helton, S. McCullough, and L. Rodman, Positive definite matrices with a given sparsity pattern, *Linear Algebra Appl.* 107:101–149 (1988).

3 J. W. Helton, S. Pierce, and L. Rodman, The ranks of extremal positive semidefinite matrices with given sparsity pattern, *SIAM J. Matrix Anal. Appl.* 10:407–423 (1989)

4 R. Grone, C. R. Johnson, E. Sa, and H. Wolkowicz, Positive definite completions of partial hermitian matrices, *Linear Algebra Appl.* 58:109–124 (1984).

5 C. Johnson and L. Rodman, Inertia possibilities for completions of partial hermitian matrices, *Linear and Multilinear Algebra* 16:179–195 (1984).

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